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Laws of Dahlquist in Multistep Methods and Some of their Modification

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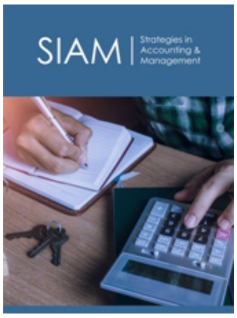

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Abstract

No one doubts the relevance Multistep Methods with constant coefficients and their applications to solve different problems of Natural science. The base search for these methods was carried out by the Dahlquist. Based on this, here the description of some development of these theories and the application to solve Ordinary Differentia the Volterra Integral and the Volterra Integro-Differential Equations. Here, investigated advanced (forward-jumping) methods, and comprised the results receiving for the advanced method with the Dahlquist results. Here, have investigated the second derivative multistep methods of advanced type and shown that stable methods of advanced methods are more exact. In addition, fined the maximum value for the degree of the advanced method. Constructed methods the illustrated receiving results.

Keywords: Initial-value problem; Ordinary differential equation; The volterra integro-differential equation; Stability and degree; Multistep multiderivative methods

Introduction

As is known in the middle ages, scientists began to study the trajectory of celestial bodies, which is usually presented as the initial-value problem for Ordinary Differential Equations. Thus, solving the above noted problem, arrives the necessity to use some numerical methods. Some authors have used the power series method for solving the named problem. L. Leonid Euler shows some disadvantages of these methods. Therefore, L. Euler constructed the new method for solving above named problems. As is known, the more general numerical methods have been constructed by Adams-Morlton and Adams-Bashforth. To illustrate the above statement, let us consider the following problem (see for example [1-11]):

$$y'(x) = f(x,y), \quad y(x_0) = y_0, x_0 \le x \le X. \quad (1)$$

Here, suppose that, this problem has the unit continuous solution y(x), which is defined in the segment $[x_0, X]$. And continuous to totality of arguments function f(x,y) has been defined in some closed set, in which has the continuous partial derivatives to some p, inclusively. As was noted above, the aim of this investigation is finding the numerical solution of the problem (1). Therefore, at the point xi, the exact value of the solution of the problem (1) have defined by the $y(x_i)$ and the corresponding approximately values by the y_i $(i = 0,1,2,..,N)$. Mesh-points x_i $(i = 0,1,.,N)$ are defended as the $x_{i+1} = x_i + h$ $(i = 0,1,..N-1)$. Here constant h-is the step-size, which divides a segment $[x_0, X]$ into N equal parts. Let us but the f define the values $f(x_i, y_i)$ $(i = 0,1,..,N)$.

Noted that Euler's method can be receive from the Adams's method as the partial case. Specialists involved in the construction and application of numerical methods to solve problem (1), generalized started all known numerical method. As the result of that the following methods were received:

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i f_{n+i},\ n = 0,1,2,..,N-k,\ \alpha_k \neq 0, \quad (2)$$

In applying this method to solving some problems of type (1), arises the question about convergence of this method. In the [1] work [12,13] has been investigated this question and proved that for the convergence of the method (2), the roots of the polynomial

$$\rho(\lambda) \equiv \alpha_k\lambda^k + \alpha_{k-1}\lambda^{k-1} + \ldots + \alpha_1\lambda + \alpha_k \quad (3)$$

must located in the unit circle on the boundary of which there is not a multiply root. This condition is called as the conception of dispersion. In the work [2], this condition is taken as the stability of the method (2). And here, have been proved that if $\beta_k = 0$ and method (2) is stable. Then $p \leq k$ for the $k \leq 10$. Here p-is the accuracy degree of the method (2). Noted that method (2) has been investigated by many authors, but fundamentally investigated by Dahlquist. Dahlquist for the study method (2), has used the conception of stability and degree, which was fined as follows:

Definition 1

The method (2) is called as the stable, if the roots of the polynomial $\rho(\lambda)$ located in the unit circle on the boundary of which there is not multiply root.

Definition 2

The integer p is called as the degree of the method (2), if the following asymptotic equalities take place,

$$\sum_{i=0}^{k}(\alpha_i y(x+ih) - h\beta_i y'(x+ih)) = O(h^{p+1}), h \to 0. \quad (4)$$

He has defined the natural conditions improved on the coefficients of the method (2).

A. The coefficients $\alpha_j, \beta_j (j = 0,1,2,..,k-m; i = 0,1,..,k)$ are some real numbers and $\alpha_{k-m} \neq 0$.

B. The characteristic polynomials

$$\rho(\lambda) \equiv \sum_{i=0}^{k-m}\alpha_i\lambda^i;\ \delta(\lambda) \equiv \sum_{i=0}^{k}\beta_i\lambda^i$$

don't have common factors different from constant.

C. The conditions $\rho(1) = 0;\ \rho'(1) = \delta(1)$ are hold.

Theorem 1

If method (2) is stable and has the degree of p, then

$$p \leq 2[k/2] + 2 \quad (5)$$

and for the each k, there are stable methods of type (2), with the degree $p_{\max} = 2[k/2] + 2$.

From the ratios (5) receive that the degree of method (2) is bounded. To increase the accuracy of the numerical methods, in the work [10] has been recommended to use the following method:

$$\sum_{i=0}^{k-m}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i f_{n+i},\ \ (n = 0,1,..,N-k,\ m < 0) \quad (6)$$

Here, suppose that $|\beta_{k-m+1}| + \ldots + |\beta_k| \neq 0.$.

Method (6) usually is called as the advanced or forward jumping method. It is obvious that, the class of methods (2) and (6) is different from each other. The next paragraph is dedicated to the study of advanced methods.

The investigation of advanced methods

The advanced methods were constructed in the beginning of XX century by Kowell (see for example [11-13]). Therefore, some authors called method (6) the Kowell method. Method (6) fully (fundamentally) has been investigated in the works (). By using the conditions $m > 0$ and $|\beta_{k-m+1}| + \ldots + |\beta_k| \neq 0$, receive that the class of method (6) is separate and independent from the method (2). Therefore, has been found the necessary conditions imposed on the coefficients of the method (6), which can be presented as following. Noted that methods of type advanced have been constructed by the famous scientists such as Laplas, Steklov, Klero, Kowelland etc. in the work [] have been constructed the concert method with the degree p=5 for the k=3. By the Dahlquist's results, receive that in the class of Multistep Method (2) there is not stable method with degree p=k+2 for the $k = 2\nu - 1$ (odd) value. By the simple comparison, the class methods (2) and (6) receive that stable methods of type (6) are more exact than the methods of type (2). For the comparison of the exactness of methods (2) and (6), let us consider the following theorem [14-18].

Theorem 2

If the method (6) is stable and has the degree of p, then the following takes place: $p \leq k + m + 1\ (k \geq 3m)$ As follows from here, the method (6) is more promising. Sometimes we come across such applied problems in the solution of which there is a need to use more accurate methods. Therefore, here is considered the construction of more exact numerical method for solving problem (1). And for this aim, Euler proposed calculations of the subsequent members of the Taylor series. For the construction more exact numerical methods for solving problem (1), Dahlquist suggest the following method (see for example [11-13],[19]):

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i f_{n+i} + h^2\sum_{i=0}^{k}\gamma_i g_{n+i};\quad n = 0,1,\ldots N-k;\ \alpha_{k-m} \neq 0. \quad (7)$$

Here the function g(x,y) defined as following: $g(x,y) = f'_x(x,y) + f'_y(x,y)f(x,y)$

Dahlquist fundamentally investigated this method and prove some theorems. The main result of the named work can be presented as following.

Theorem 3

Suppose that method of (7) is stable and has the degree of p, then

$$p \leq 2k + 2f\ \ |\beta_k| + |\gamma_k| \neq 0 \text{ and p=2k } \beta_k = \gamma_k = 0. \quad (8)$$

For the each k, there exists stable method of type (7) with the degree of $p = 2k + 2$.

To increase the accuracy of the method (7), in [12] the method (7) has been modified as the following form:

$$\sum_{i=0}^{k-m}\alpha_i y_{n+i} = h\sum_{i=0}^{l}\beta_i f_{n+i} + h^2\sum_{i=0}^{s}\gamma_i g_{n+i};\quad n = 0,1,\ldots N-k;\ \alpha_{k-m} \neq 0,\ m > 0. \quad (9)$$

Method (7) can be received from the method (9) the as partial case.

The method fully explored in the work [18]. Note that the case $\max(l,s) \le k-m$, investigated by Dahlquist. Therefore, here consider to investigation the method (9) in the case $\max(l,s) > k-m$. Let us consider the definition of the maximum order of accuracy for the method (9) in the case $\max(l,s) > k-m$.

Theorem 4

If method (9) has degree of p and stable in the case $\max(l,s) > k-m$, then there are stable methods of type (9) with the degree $p_{\max} = l+s+m+2$ $(k \ge 3m)$. Note that if in the method (7) the coefficients $\beta_i (i=0,1,..,k)$ satisfy the condition $\beta_i = 0$ $(i=0,1,..,k)$, then the definition of stability changes dramatically [20-39]. In this case, the conception stability can be define as the following:

Definition 3

Method (7) called as the stable in the case $\beta_i = 0$ $(i=0,1,..,k)$, if the rots of the polynomial $\rho(\lambda) \equiv \sum_{i=0}^{k} \alpha_i \lambda^i$ located in the unit circle on the boundary of which there is not multiply roots with the exception of the double root $\lambda = 1$.

For the illustration of the results given above let us to consider the following methods of type (9):

$$y_{n+3} = (y_{n+2} + y_{n+1} + y_n)/3 + h(10781 f_{n+3} + 22707 f_{n+2} + 16659 f_{n+1} + 4285 f_n)/27216 + h^2(-2099 g_{n+3} + 7227 g_{n+1} + 979 g_n)/45360 (p=9) \quad (10)$$

$$y_{n+2} = (416 y_{n+1} - 103 y_n)/313 + h(157 f_{n+3} + 11233 f_{n+2} + 8521 f_{n+1} - 2830 f_n)/25353 + h^2(-11 g_{n+3} + 630 g_{n+2} + 1557 g_{n+1} - 92 g_n)/8451 (p=9) \quad (11)$$

$$y_{n+1} = y_n + h(1985 f_{n+3} + 12015 f_{n+2} + 142255 f_{n+1} + + 34465 f_n)/90720 + h^2(-163 g_{n+3} + 2421 g_{n+2} + 7659 g_{n+1} + 1283 g_n)/30240 \quad (12)$$

By the simple comparison, receive that the properties of these methods are subject to the above-obtained results of the law. The similarly results one can be fined in the works (see for example [40-49]).

Numerical Results

For the illustration of the results about the contraction the new way to receive more that result, let us applied method of type (2) to solve following very simple problem:

$y' = \cos x$, $y(0) = 0$, $0 \le x \le 1$, exact solution for which has define as the y(x)=sinx.

For solving this example, let us to use the following Simpson method:

$$y_{n+1} = y_n + h(f_n + 4 f_{n+1} + f_{n+2})/3$$

and the following Simpson method with the step-size

$$y_{n+2} = y_n + h(f_n + 4 f_{n+1/2} + f_{n+2})/3$$

Results receiving for these methods have tabulated in the Table 1.

Table 1: Results for the step size h: =0.1

Step Size	Variable x	Error for the Simpson's Method	Error for the Method (*)
h=0.1	0.2	0.11E-06	0.69E-08
	0.3	0.10E-06	0.10E-07
	0.4	0.21E-06	0.13E-07
	0.5	0.21E-06	0.16E-07
	0.6	0.31E-06	0.19E-07
	0.7	0.30E-06	0.22E-07
	0.8	0.39E-06	0.24E-07
	0.9	0.38E-06	0.27E-07
	1.0	0.46E-06	0.29E-07

Conclusion

Here consider a comparison of some results for Multistep Methods with the constant coefficients using first and second derivatives solutions to the problem (1). Have shown that, there are some class of methods using as the first and second derivative of the solution of problem (1). And this class methods generalizes of the known Multistep Second Derivative Methods with constant coefficients. Here is a complete comparison of methods using the first derivative of the solution of the problem (1). And have shown that the new class method of type (6) different class methods. By simple comparison get those methods like type objectivity, let us note that the advanced methods have some disadvantages. For example, to define the values y_{n+k-m} by the advanced methods it is necessary that the values of the sought solutions at subsequent points must be known. Note that for solving this problem one can use the predictor-corrector method. By using the predictor-corrector method one can be expanded the region of stability for the stable advanced methods. Here described the application of the Multistep Methods to solve initial-value problem for Ordinary Differential Equations. However, this method can be applied with equal success to solve other problems. For example the initial-value problem for the Volterra Integral Differential Equations. To verify this it is enough to write the method (2) in the following from:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i y'_{n+i}, \ n = 0,1,\ldots,N-k, \ \alpha_k \ne 0.$$

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Conflict of Interest

The authors state express that there is no conflict of interest misunderstanding between them. We hereby confirm that all the methods in this manuscript are ours.

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