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Research Article

A SIZE- BIASED DISCRETE AKASH DISTRIBUTION WITH PROPERTIES AND ITS APPLICATIONS

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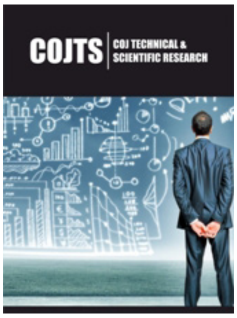

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Abstract

A size-biased discrete Akash distribution has been proposed. The raw moments and central moments have been derived and hence expressions for coefficient of variation, skewness, kurtosis and index of dispersion have been given. The estimation of its parameter has been discussed. Four examples of observed real datasets regarding distribution of freely forming group size have been presented to test the goodness of fit of the proposed distribution over size-biased Poisson distribution, size-biased Poisson-Lindley distribution, size-biased Poisson-Akash distribution and size-biased discrete Lindley distribution.

Keywords: Size-biased distribution; Discrete-akash distribution; Moments and moments-based measures; Estimation of parameter; Goodness of fit

Introduction

Size-biased distributions are a case of weighted distributions which arise naturally in practice when observations from a sample are recorded with probability proportional to some measure of unit size. In field applications, size-biased distributions can arise either because individuals are sampled with unequal probability by design or because of unequal detection probability. Size-biased distributions come into play when organisms occur in groups, and group size influences the probability of detection. Fisher [1] firstly introduced these distributions to model ascertainment biases which were later reformulated by Rao [2] in a unifying theory for problems where the observations fall in non-experimental, non-replicated and non-random categories. Size-biased distributions have applications in almost every fields of knowledge including environmental science, econometrics, social science, biomedical science, human demography, ecology, geology, forestry etc. Further, size-biasing occurs in many unexpected contexts such as statistical estimation, renewal theory, infinite divisibility of distributions and number theory. Van Duesen [3] has detailed study about the applications of size-biased distributions for fitting distributions of diameter at breast height (DBH) data arising from horizontal point sampling (HPS). Later, Lappi J et al. [4] have applied size-biased distributions to analyze HPS diameter increment data. The applications of size-biased distributions to the analysis of data relating to human population and ecology can be found in Patil & Rao [5,6].

Let a random variable has probability distribution $P_0(x;\theta)\,;x=0,1,2,...,\theta>0$. If sample units are weighted or selected with probability proportional to x^{α}, then the corresponding size-biased distribution of order α is given by its probability mass function (pmf)

$$P_1(x;\theta)=\frac{x^{\alpha}\cdot P_0(x;\theta)}{\mu'_{\alpha}} \quad (1.1)$$

Where $\mu'_{\alpha}=E(X^{\alpha})=\sum_{x=0}^{\infty}x^{\alpha}P_0(x;\theta)$. The distribution is known as size-biased or area biased according as $\alpha=1$ or $\alpha=2$ respectively.

Shanker [7] introduced one parameter Akash distribution defined by probability density function (pdf) and cumulative distribution function (cdf)

$$f_1(x;\theta)=\frac{\theta^3}{\theta^2+2}\left(1+x^2\right)e^{-\theta x}\quad ;x>0,\ \theta>0 \quad (1.2)$$

$$F_1(x;\theta)=1-\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x}\quad ;x>0,\ \theta>0 \quad (1.3)$$

Shanker [7] has discussed its various interesting and important statistical properties, estimation of parameter and applications.

Berhane et al. [8] introduced a discrete-Akash distribution (DAD), discrete form of Akash distribution using infinite series approach of discretization having pmf

$$P_2(x;\theta)=\frac{\left(e^{\theta}-1\right)^3}{e^{\theta}\left(e^{2\theta}-e^{\theta}+2\right)}\left(1+x^2\right)e^{-\theta x};\ x=0,1,2,...,\theta>0 \quad (1.4)$$

Various statistical properties of DAD, estimation of parameter and applications to model count data have been studied by Berhane [8] and it has been observed that it gives better fit than both Poisson distribution (PD) and Poisson-Lindley distribution (PLD), a Poisson mixture of Lindley [9] distribution and introduced by Sankaran [10]. The first four moments about origin and the variance of DAD obtained by Berhane et al. [8] are given by

$$\mu_1'=\frac{2\left(e^{2\theta}+e^{\theta}+1\right)}{\left(e^{2\theta}-e^{\theta}+2\right)\left(e^{\theta}-1\right)}$$

$$\mu_2'=\frac{2\left(e^{3\theta}+5e^{2\theta}+5e^{\theta}+1\right)}{\left(e^{2\theta}-e^{\theta}+2\right)\left(e^{\theta}-1\right)^2}$$

$$\mu_3'=\frac{2\left(e^{4\theta}+14e^{3\theta}+30e^{2\theta}+14e^{\theta}+1\right)}{\left(e^{2\theta}-e^{\theta}+2\right)\left(e^{\theta}-1\right)^3}$$

$$\mu_4'=\frac{2\left(e^{5\theta}+33e^{4\theta}+146e^{3\theta}+146e^{2\theta}+33e^{\theta}+1\right)}{\left(e^{2\theta}-e^{\theta}+2\right)\left(e^{\theta}-1\right)^4}$$

$$\mu_2=\frac{2e^{\theta}\left(e^{4\theta}+2e^{3\theta}-2e^{2\theta}+5\right)}{\left(e^{2\theta}-e^{\theta}+2\right)^2\left(e^{\theta}-1\right)^2}$$

Simon et al. [11] introduced a size-biased discrete Lindley distribution (SBDLD) defined by its pmf

$$P_3(x;\theta)=\frac{\left(e^{\theta}-1\right)^3}{2e^{2\theta}}\left(x+x^2\right)e^{-\theta x}\ ;x=1,2,3,...,\theta>0 \quad (1.5)$$

Statistical properties, estimation of parameter using both the method of moment and the method of maximum likelihood and the applications of SBDLD are available in Simon et al. [11] Note that a discrete Lindley distribution (DLD) using infinite series approach of discretization has been proposed by Berhane and Shanker [12] from Lindley distribution and its properties, estimation and applications are available in Shanker et al. [12].

It has been observed that DAD gives much closer fit over DLD, PLD and Poisson distribution, it is expected that size-biased discrete Akash distribution (SBDAD would also give much better fit over SBPD, SBPLD and SBDLD. Keeping these points in mind, SBDAD has been proposed by size-biasing the discrete-Akash distribution (DAD) of Berhane et al. [8], a discrete form of continuous Akash distribution introduced by Shanker [7]. Its moments and moments-based measures including coefficient of variation, skewness, kurtosis and index of dispersion have been derived. The estimation of its parameter has been discussed using the method of moment and the method of maximum likelihood. Four examples of observed real datasets have been presented to test the goodness of fit of SBDAD over size-biased Poisson distribution (SBPD), size-biased Poisson-Lindley distribution (SBPLD), size-biased Poisson-Akash distribution (SBPAD) and size-biased discrete Lindley distribution (SBDLD).

Size- Biased Discrete-Akash Distribution

Using (1.1) and (1.4) and the expression for the mean of DAD, the size-biased discrete-Akash distribution (SBDAD) with parameter θ can be defined by its pmf

$$P_4(x;\theta)=\frac{x\cdot P_0(x;\theta)}{\mu_1'}=\frac{(e^{\theta}-1)^4}{2e^{\theta}(e^{2\theta}+e^{\theta}+1)}(x+x^3)e^{-\theta x}\quad ;\quad x=1,2,3,..,\theta>0 \quad (2.1)$$

It can be easily verified that SBDAD is unimodal and have increasing failure rate. Since

$$\frac{P_4(x+1;\theta)}{P_4(x;\theta)}=\left(\frac{1}{e^{\theta}}\right)\left(1+\frac{3x^2+3x+2}{x+x^3}\right)$$

is a deceasing function of x, $P_4(x;\theta)$, is log-concave. Therefore, SBDAD is unimodal, has an increasing failure rate (IFR), and hence increasing failure rate average (IFRA). It is new better than used in expectation (NBUE) and has decreasing mean residual life (DMRL). The definitions and relationship between these aging concepts have been discussed in Barlow [13]. The graphs of the pmf of SBDAD (2.1) for varying values of the parameter (Figure 1).

The size-biased Poisson-Lindley distribution (SBPLD) introduced by Ghitany et al. [13] is defined by its pmf

$$P_5(x;\theta)=\frac{\theta^3}{\theta+2}\frac{x(x+\theta+2)}{(\theta+1)^{x+2}}\ ;x=1,2,3,...,;\theta>0 \quad (2.5)$$

It should be noted that SBPLD is a size-biased version of Poisson-Lindley distribution (PLD) introduced by Sankaran [10]. Ghitany et al. [14] have discussed its various mathematical and statistical properties, estimation of the parameter using maximum likelihood estimation and the method of moments, and goodness of fit. Shanker et al. [15] has critical study on the applications of SBPLD for modeling data on thunderstorms and found that SBPLD is a better model for thunderstorms than size-biased Poisson distribution (SBPD). The size-biased Poisson-Akash distribution (SBPAD) proposed by Shanker [16] is defined by its pmf

$$P_6(x;\theta)=\frac{\theta^4}{\theta^2+6}\frac{\left\{x^3+3x^2+\left(\theta^2+2\theta+3\right)x\right\}}{\left(\theta+1\right)^{x+3}}\quad ;x=1,2,3,..,\theta>0 \tag{2.6}$$

Note that SBPAD is a size-biased version of Poisson-Akash distribution (PAD) suggested by Shanker [17]. Shanker has detailed study on various statistical properties, estimation of parameter and applications of both PAD and SBPAD.

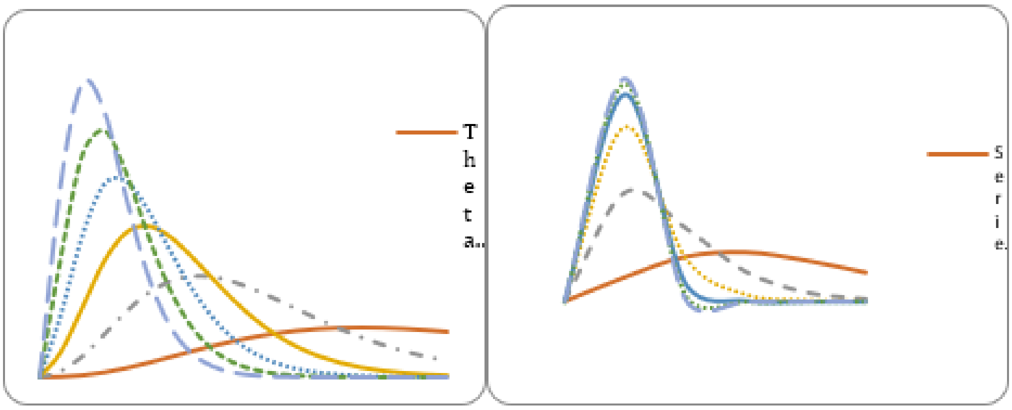

Figure 1: Graphs of pmf of SBDAD for varying values of the parameter θ.

Moments and Moments Based Measures

Using (2.1), the r^{th} moment of origin μ_r' of the SBDAD (2.1) can be obtained as

$$\mu_r'=E\left(X^r\mid\theta\right)$$

$$=\left[\sum_{x=1}^{\infty}x^r\frac{(e^\theta-1)^4}{2e^\theta(e^{2\theta}+e^\theta+1)}(x+x^3)e^{-\theta x}\right]$$

$$=\frac{(e^\theta-1)^4}{2e^\theta(e^{2\theta}+e^\theta+1)}\sum_{x=1}^{\infty}(x^{r+1}+x^{r+3})e^{-\theta x};r=1,2,3,... \tag{3.1}$$

Taking in (3.1), the first four moments about origin of the SBDAD (2.1) are thus obtained as

$$\mu_1'=\frac{e^{3\theta}+5e^{2\theta}+5e^\theta+1}{(e^\theta-1)(e^{2\theta}+e^\theta+1)}$$

$$\mu_2'=\frac{e^{4\theta}+14e^{3\theta}+30e^{2\theta}+14e^\theta+1}{(e^\theta-1)^2(e^{2\theta}+e^\theta+1)}$$

$$\mu_3'=\frac{e^{5\theta}+33e^{4\theta}+146e^{3\theta}+146e^{2\theta}+33e^\theta+1}{(e^\theta-1)^3(e^{2\theta}+e^\theta+1)}$$

$$\mu_4'=\frac{e^{6\theta}+72e^{5\theta}+603e^{4\theta}+1168e^{3\theta}+603e^{2\theta}+72e^\theta+1}{(e^\theta-1)^4(e^{2\theta}+e^\theta+1)}$$

Now, using the relationship between moments about mean and the moments about origin, the moments about mean of the SBDAD (2.1) can be obtained as

$$\mu_2=\sigma^2=\frac{5e^{4\theta}+10e^{3\theta}+6e^{2\theta}+10e^\theta+1}{(e^\theta-1)^2(e^{2\theta}+e^\theta+1)^2}$$

$$\mu_3=\frac{e^\theta(5e^{7\theta}+20e^{6\theta}+18e^{5\theta}+65e^{4\theta}+65e^{3\theta}+18e^{2\theta}+20e^\theta+1)}{(e^\theta-1)^3(e^{2\theta}+e^\theta+1)^3}$$

$$\mu_4=\frac{e^\theta(5e^{10\theta}+115e^{9\theta}+354e^{8\theta}+780e^{7\theta}+1125e^{6\theta}+1074e^{5\theta}+1125e^{4\theta}+780e^{3\theta}+354e^{2\theta}+115e^\theta+1)}{(e^\theta-1)^4(e^{2\theta}+e^\theta+1)^4}$$

The coefficient of variation $(C.V)$, coefficient of Skewness $\left(\sqrt{\beta_1}\right)$, coefficient of Kurtosis (β_2) and index of dispersion (γ) of the SBDAD (2.1) are thus obtained as

$$C.V=\frac{\sigma}{\mu_1'}=\frac{\sqrt{e^\theta\left(5e^{4\theta}+10e^{3\theta}+6e^{2\theta}+10e^\theta+5\right)}}{\left(e^{3\theta}+5e^{2\theta}+5e^\theta+1\right)}$$

$$\sqrt{\beta_1}=\frac{\mu_3}{\mu_2^{3/2}}=\frac{e^\theta(e^{7\theta}+20e^{6\theta}+18e^{5\theta}+65e^{4\theta}+65e^{3\theta}+18e^{2\theta}+20e^\theta+5)}{\left(5e^{4\theta}+10e^{3\theta}+6e^{2\theta}+10e^\theta+5\right)^{3/2}}$$

$$\beta_2=\frac{\mu_4}{\mu_2^2}=\frac{\left(e^{10\theta}+115e^{9\theta}+354e^{8\theta}+780e^{7\theta}+1125e^{6\theta}+1074e^{5\theta}+1125e^{4\theta}+780e^{3\theta}+354e^{2\theta}+115e^\theta+5\right)}{e^\theta\left(5e^{4\theta}+10e^{3\theta}+6e^{2\theta}+10e^\theta+5\right)^2}$$

$$\gamma=\frac{\sigma^2}{\mu_1'}=\frac{\left(5e^{4\theta}+10e^{3\theta}+6e^{2\theta}+10e^\theta+5\right)}{(e^\theta-1)\left(e^{2\theta}+e^\theta+1\right)\left(e^{3\theta}+5e^{2\theta}+5e^\theta+1\right)}$$

The graphs of coefficient of variation $(C.V)$, coefficient of Skewness $\left(\sqrt{\beta_1}\right)$, coefficient of Kurtosis (β_2) and index of dispersion (γ) of the SBDAD are shown in (Figure 2). It is obvious that C.V and index of dispersion are monotonically decreasing whereas coefficient of skewness and coefficient of kurtosis are monotonically increasing for increasing values of the parameter θ.

It can be easily verified that SBDAD is over-dispersed $\left(\mu<\sigma^2\right)$, equi-dispersed $\left(\mu=\sigma^2\right)$ and under-dispersed for $\theta<(=)>\theta^*=0.583686$. It should be noted that SBDLD is over-dispersed $\left(\mu<\sigma^2\right)$, equi-dispersed $\left(\mu=\sigma^2\right)$ and under-dispersed for $\theta<(=)>\theta^*=1.00505$.

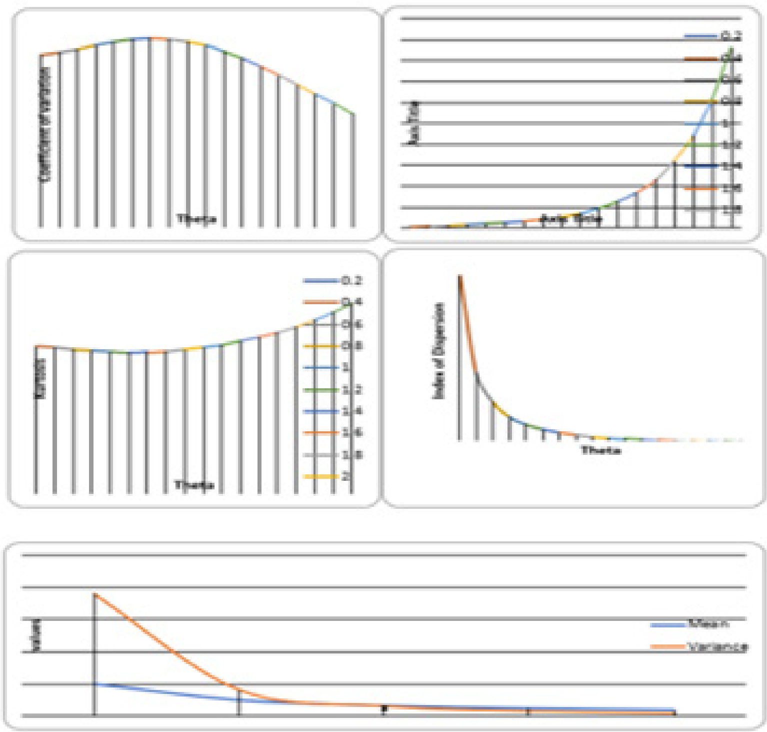

Figure 2: Graphs of C.V, coefficient of Skewness, coefficient of Kurtosis and index of dispersion of the SBDAD for varying values of the parameter θ.

Parameter Estimation

Method of moment estimate (MOME): Equating the population mean to the corresponding sample mean, the method of moment estimate (MOME) $\tilde{\theta}$ of θ of SBDAD is the solution of the following non-linear equation in θ

$(1-\bar{x})e^{3\theta}+5e^{2\theta}+5e^{\theta}+(1+\bar{x})=0$, where is the sample mean.

Maximum likelihood estimate (MLE): Let $x_1, x_2, ..., x_n$ be a random sample of size from the SBDAD (2.1) and let f_x be the observed frequency in the sample corresponding to $X=x\ (x=1,2,3,...,k)$ such that $\sum_{x=1}^{k} f_x = n$, where is the largest observed value having non-zero frequency. The likelihood function L of the SBDAD (2.1) is given by

$$L=\left(\frac{(e^{\theta}-1)^4}{2e^{\theta}(e^{2\theta}+e^{\theta}+1)}\right)^n e^{-\theta\sum_{x=1}^{k} x f_x} \prod_{x=1}^{k}\left(x+x^3\right)^{f_x}$$

The log likelihood function can be obtained as

$$\log L = n\log\left(\frac{(e^{\theta}-1)^4}{2e^{\theta}(e^{2\theta}+e^{\theta}+1)}\right)-\theta\sum_{x=1}^{k} x f_x+\sum_{i=1}^{k} f_x \log\left(x+x^3\right)$$

The first derivative of the log likelihood function is thus given by

$$\frac{d\log L}{d\theta}=\frac{4ne^{\theta}}{\left(e^{\theta}-1\right)}-\frac{n\left(3e^{3\theta}+2e^{2\theta}+e^{\theta}\right)}{e^{3\theta}+e^{2\theta}+e^{\theta}}-n\bar{x}$$

where $\bar{x}$ is the sample mean

The maximum likelihood estimate (MLE), $\hat{\theta}$ of θ of SBDAD (2.1) is the solution of the equation $\frac{d\log L}{d\theta}=0$ and is given by the solution of the following non-linear equation

$$\frac{d\log L}{d\theta}=\frac{4e^{\theta}}{\left(e^{\theta}-1\right)}-\frac{\left(3e^{3\theta}+2e^{2\theta}+e^{\theta}\right)}{e^{3\theta}+e^{2\theta}+e^{\theta}}-\bar{x}=0$$

This gives $(1-\bar{x})e^{3\theta}+5e^{2\theta}+5e^{\theta}+(1+\bar{x})=0$. Thus, both the method of moment and the method of maximum likelihood gives the same equation for estimating the parameter θ. This non-linear equation can be solved by any numerical iteration methods such as Newton- Raphson method, Bisection method, Regula-Falsi method etc. In the present paper, Newton-Raphson method has been used.

Goodness of Fit

As we know that size-biased distributions are used to model data when organism occur in groups and the size of the group influence the probability of detection. Here we considered four datasets relating to size-distribution of freely forming small groups at various public places, available in Coleman et al. [18]. The goodness of fit of SBPD, SBPLD, SBPAD, SBDLD and SBDAD has been presented for four count datasets (Table 1-3).

It is obvious from the goodness of fit in these tables that SBDAD gives much closer fit than other size-biased distribution except in (Table 4), where SBDLD gives better fit over other size-biased distributions. Therefore, SBDAD can be considered as a model for size-distribution of freely forming small groups.

Table 1: Pedestrians-Eugene, Spring, Morning.

Group Size	Observed Frequency	Expected Frequency				
		SBPD	SBPLD	SBPAD	SBDLD	SBDAD
1	1486	1452.4	1532.5	1539.76	1510.58	1486.43
2	694	743.3	630.6	620.54	660.4	693.05
3	195	190.2	191.9	191.45	192.48	193.88
4	37	32.4	51.3	52.99	46.75	41.98
5	10	4.1	12.8	13.77	10.22	7.31
6	1	0.6	3.9	3.42	2.57	1.5
Total	2423	2423	2423	2423	2423	2423
ML estimate	$\hat{\theta} = 0.5118$	$\hat{\theta} = 4.5082$		4.7295	1.926	$\hat{\theta} = 2.3725$
χ^2		7.37	13.824	17.19	4.426	1.143
d.f.		2	3	3	3	3
p-value		0.06097	0.00788	0.00142	0.3514	0.8874

Table 2: Shopping Groups-Eugene, Spring, Department store and Public Market.

Group Size	Observed Frequency	Expected Frequency				
		SBPD	SBPLD	SBPAD	SBDLD	SBDAD
1	316	306.3	323	324.62	318.48	313.43
2	141	156.2	132.5	130.41	138.78	145.6
3	44	39.8	40.2	40.11	40.32	40.62
4	5	6.8	10.7	11.06	9.76	8.55
5	4	0.9	3.6	3.8	2.66	1.9
Total	510	510	510	510	510	510
ML estimate	$\hat{\theta} = 0.5098$	$\hat{\theta} = 4.5224$		4.7446	1.9293	$\hat{\theta} = 2.3759$
χ^2		2.449	3.02053	3.777	1.332	0.64885
d.f.		2	2	2	2	2
p-value		0.4846	0.3884	0.2866	0.7214	0.8851

Table 3: Play Groups-Eugene, Spring, Public Playground D.

Group Size	Observed Frequency	Expected Frequency				
		SBPD	SBPLD	SBPAD	SBDLD	SBDAD
1	305	296.5	314.4	315.93	309.5	304.09
2	144	159	134.4	132.19	140.85	148.02
3	50	42.6	42.5	42.34	42.73	43.23
4	5	7.6	11.8	12.15	10.8	9.54
5	2	1	3.1	3.27	2.46	1.78
6	1	0.3	0.8	1.12	0.66	0.34
Total	507	507	507	507	507	507
ML estimate	$\hat{\theta} = 0.5365$	$\hat{\theta} = 4.3179$		4.557304	1.8858	$\hat{\theta} = 2.3294$
χ^2		3.035	6.0667	7.228	3.891	2.32096
d.f.		2	2	2	2	2
p-value		0.3862	0.1084	0.065	0.2735	0.5085

Table 4: Play Groups-Eugene, Spring, Public Playground A.

No. Times Hares Caught	Observed Frequency	Expected Frequency				
		SBPD	SBPLD	SBPAD	SBDLD	SBDAD
1	306	292.2	309.4	311	304.76	299.53
2	132	155.2	131.2	129.11	137.53	144.49
3	47	41.2	41.1	41.02	41.38	41.82
4	10	7.3	11.3	11.69	10.37	9.14
5	2	1.1	4	4.08	2.45	2.02
Total	497	497	497	497	497	497
ML estimate	$\hat{\theta} = 0.5312$	$\hat{\theta} = 4.3548$		4.5931	1.8943	$\hat{\theta} = 2.3385$
χ^2		6.479	1.60096	1.918	1.043	1.924
d.f.		2	2	2	2	2
p-value		0.09048	0.6592	0.5896	0.7908	0.5882

Concluding Remarks

In the present paper size-biased Discrete-Akash distribution (SBPAD), a simple size-biased version of the discrete-Akash distribution (DAD) of Berhane et al. [8] has been proposed and studied. Its raw moments and central moments have been obtained and hence expressions for coefficient of variation (C.V.), skewness, kurtosis and index of dispersion have been presented and their natures have been discussed graphically. The estimation of its parameter has been discussed using the method of moments and the method of maximum likelihood estimation. The goodness of fit of the SBDAD has been discussed with four examples of observed real datasets over SBPD, SBPLD, SBPAD, SBDLD and it is observed that SBDAD gives much closer fit. Therefore, SBDAD can be considered an important distribution for modeling distribution of freely forming group size.

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