Crimson Publishers
Wings to the Research

Mini Review

Reduced (Twisted) Group C*-Algebras without Nontrivial Ideals

Mingchu Gao*

Department of Mathematics, USA

ISSN: 2640-9739

***¹Corresponding author:** Mingchu Gao, Department of Mathematics, Pineville, LA 71360, USA

Submission: August 03, 2022

Published: February 03, 2023

Volume 2 - Issue 4

How to cite this article: Mingchu Gao*. Reduced (Twisted) Group C*-Algebras without Nontrivial Ideals. COJ Elec Communicat. 2(4).COJEC.000544.2023. DOI: 10.31031/ COJEC.2023.02.000544

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Historical Background

The class of group operator algebras was a typical model Murray and von Neumann studied in initiating the theory of operator algebras [1-8]. Since then, the interplay between groups and operator algebras has been a main line in the development of operator algebras. In this article, we review the recent work on determining when a countable discrete group is C^*-simple, i.e., its reduced group C^*algebra has no nontrivial closed two side ideas. We also discuss the question for twisted group algebras. All groups in this article are assumed to be countable and discrete. Let G be a group.

The left regular unitary representation $\lambda : G \rightarrow B(l_2(G))$ is defined by $(\lambda(g)\xi)(h) = \xi(g-1h)$, for all g, $h \in G$ and $\xi \in l2(G)$. The C^*-algebra generated by $\{\lambda(g) : g \in G\}$ is the reduced group C^*-algebra of G denoted by $C^*_r(G)$.

In 1949, I. Kaplansky asked R. Kadison whether any simple unital C^*-algebra other than C has a nontrivial projection. In 1968, Kadison suggested R. Powers to study from this point of view the reduced group C^* algebra $C^*_r(F_2)$ of the non-abelian free group with two generators. Powers showed within a week that is simple and published the work several years later [9,10]. Since then, considerable efforts have been made in finding C^*-simple groups. The generalization/modification of Powers' proof had been the only method in finding C^*-simple groups until M. Kalantar and M. Kennedy's breakthrough work [6].

Recent Prograss on C*-Simple Groups

Recall that an action of a group G on a compact Hausdorff space X is said to be strongly proximal if for each probability measure μ on X, the week ∗-closure of the orbit $G.\mu$ contains a point-mass δ_x, for some $x \in X$. An action $G \curvearrowright X$ is a boundary action if it is strongly proximal and minimal. In this case, we call X a G-boundary. Recall also that the amenable radical Rad(G) of a group G is defined as the largest normal amenable subgroup of G.

The following Furman's result gives the existence of boundary actions of a group.

Theorem 0.1 [4]

Let G be a group and $t \in G$. Then t /∈ Rad(G) if and only if there is a G-boundary X such that t acts non-trivially on X.

An action $G \curvearrowright X$ is free if $X_g = \{x \in X : gx = x\} = \emptyset$ for every non-identity $g \in G$. An action $G \curvearrowright X$ is topologically free if $X_g = \{x \in X : gx = x\}$ has an empty interior for every non-identity element $g \in G$. Kalantar and Kennedy proved in [6] that a discrete group G is C^*-simple if and only if G acts topologically freely on some G-boundary. By Proposition 2.5 in [3], the action of G on its universal boundary $\partial_F G$ is free if it is topologically free. Hence, we have the following characterization of C^*-simple groups.

Theorem 0.2 [3]

A group G is C^*-simple if and only if G acts freely on some G-boundary X. A subgroup H of group G is recurrent if there is a finite subset $F \subseteq G\setminus\{e\}$ such that $F \cap gHg^{-1} = \emptyset,\ \forall g \in G$. Kennedy [7] obtained the following intrinsic characterization of C^*-simple groups.

Theorem 0.3 [7]

A group is C^*-simple if and only if it has no amenable recurrent subgroups.

U. Haagrep [5] characterized C^*-simple groups in terms of Dixmier-type properties.

Theorem 0.4 [5]

Let G be a group. Then G is C^*-simple if and only if for all $t_1,\ldots,t_m \in G \setminus \{e\}$,

$$0 \in \overline{conv}\{\lambda(s)(\lambda(t1) + \ldots + \lambda(t_m))\lambda(s)^* : s \in G\},$$

Where $\overline{conv}$ is the closure of all convex combinations of the elements in the set.

Twisted Group C*-Algebras

The theory of twisted group C^*-algebras is closed related to projective unitary representations of groups with important applications in various fields of mathematics and physics [9].

Let G be a group and $\pi : G \to U(H)$, where $U(H)$ is the unitary group of Hilbert space H. We say that π is a projective unitary representation of G if

(1) $$\pi(g)\pi(h) = \sigma(g,h)\pi(gh), \forall g,h \in G,$$

where $\sigma : G \to T$ is a function on G and $T = \{z \in C : |z| = 1\}$. From (1), we get

(2) $$\sigma(g,e) = \sigma(e,g) = 1, \sigma(g,h)\sigma(gh,k) = \sigma(g,hk)\sigma(h,k), \forall g,h,k \in G.$$

A function $\sigma : G \to T$ is called a 2-cycocle on G if it satisfies (2). The above described representation π is called a σ-projective unitary representation of G. We define $\lambda\sigma : G \to U(l^2(G))$ by $(\lambda_\sigma(g)\xi)(h) = \sigma(g,g^{-1}h)\xi(g^{-1}h),$, for g, $h \in G$ and $\xi \in l^2(G)$. This representation is called the left regular σ-projective unitary representation of G. The C^*-algebra generated by $\{\lambda\sigma(g) : g \in G\}\ in\ B(l^2(G))$ is called the reduced twisted group C^*-algebra of $C^*_T(G,\sigma)$.

Let $\pi : G \to U(H)$ be a σ-projective unitary representation of G and $\xi \in H$. The map $\varphi : g \mapsto \langle \pi(g)\xi, \xi \rangle$ is called a diagonal matrix coefficient of π. Given two σ-projective unitary representations π and ρ of a group G, say that π is weakly contained in ρ, write $\pi \prec \rho$, if any diagonal matrix coefficient of π is a limit of sums of diagonal matrix coefficients of ρ, uniformly on every finite subsets of F. We say that π is weakly equivalent to ρ, write $\pi \sim \rho$, if $\pi \prec \rho$ and $\rho \prec \pi$.

Determining when $C_r^*(G,\sigma)$ is simple a very popular question in operator algebras. There are many discussions on this topic. For instance, Bedos and Omland [2] gave some sufficent conditions for $C_r^*(G,\sigma)$ be to simple. They also applied their results to different types of groups such as wreath products and Baumslag-Solitar groups. Very recently, we used weak containment of projective unitary representations to give a characterization of the simplicity of $C_r^*(G,\sigma)$

Theorem 0.5 [1]

The algebra $C_r^*(G,\sigma)$ is simple if and only if for every σ-projective unitary representation π of G, if $\pi \prec \lambda_\sigma$ then $\pi \sim \lambda_\sigma$.

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