

A Probabilistic Bound on the Distribution of Primes in Consecutive Intervals of Natural Numbers

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Abstract

This research investigates the upper bound on the probability of encountering prime numbers within any set of 100 consecutive natural numbers. Specifically, it is shown that this probability cannot exceed 0.25, implying that at most 25 primes exist in such an interval. Using a combination of analytical reasoning based on the prime number theorem and a mathematical induction framework, we rigorously establish this result. The study further examines the transition between overlapping consecutive intervals, accounting for the inclusion or exclusion of boundary numbers and confirms the invariance of the upper bound. This result provides a deeper understanding of the density of primes in finite intervals and offers insights into their distribution properties.

Keywords: Prime number theorem; Prime counting function; Mathematical induction; Probability; Natural numbers

Introduction

Prime numbers are fundamental objects in number theory, with applications in cryptography, mathematical modelling and theoretical computer science. One long-standing area of interest is understanding the distribution of primes within finite intervals. The prime number theorem provides an asymptotic approximation for the distribution of primes up to a large number x , yet exact counts in smaller intervals often require more detailed analysis. This research focuses on a specific probabilistic bound: the likelihood of encountering a prime number in any set of 100 consecutive natural numbers. Empirical observation suggests that the number of primes in such intervals seldom exceeds 25, motivating the hypothesis that the probability of a number being prime in such an interval is bounded by 0.25. To formalize this observation, we analyze the problem through two complementary approaches:

- A. Analytical reasoning using the prime number theorem: This approach estimates the density of primes in small intervals.
- B. Proof by mathematical induction: This technique rigorously confirms the bounded behavior of the number of primes when transitioning between overlapping intervals of consecutive numbers.

By combining these approaches, we prove that the probability of finding a prime in any set of 100 consecutive numbers does not exceed 0.25.

Mathematical Preliminaries and Prime Number Theorem

The Prime Number Theorem (PNT) describes the asymptotic distribution of prime numbers [1]. It states that the number of prime numbers less than or equal to a given number x , denoted by $\pi(x)$ and called prime counting function [2,3], is approximately given by the formula

$$\delta(x) = \frac{x}{\ln x} \quad (1)$$

Where, $\ln x$ is the natural logarithm of x . This theorem provides a fundamental understanding of how primes are distributed among the natural numbers. This asymptotic result guides our understanding of the average distribution of primes but does not directly answer questions about finite intervals. For smaller intervals, we rely on precise calculations of $\pi(x)$ and logical reasoning to bound the number of primes.

Problem Statement

The numbers of prime number in the set $A = \{n, n + 1, \dots, n + 99: n \in \mathbb{N}\}$ cannot be more than 25. In other words, if p denotes the probability of a prime number choosing from A than $p \leq \frac{1}{4}$.

To prove the statement using mathematical induction, we show that:

Base Case: The result holds true for $n = k$, where k is a positive integer.

Inductive Step: If the result holds true for $n=k$, we prove it also holds for $n = k+1$.

The statement we are proving is:

The probability of finding a prime in any set of 100 consecutive natural numbers is at most 0.25. This means:

$$\Pi(k + 100) - \pi(k) \leq 25 \quad (2)$$

where, $\pi(x)$ is the prime-counting function.

Step 1: Base Case

For $n = 1$ the interval $[1,100]$. We calculate the number of primes in this interval. The primes in this range are: 2,3,5,7,11,13,17,19,23,29,31,37,41,43,47,53,59,61,67,71,73,79,83,89,97.

There are 25 primes in total.

Thus, the probability is: $P = 25/100 = 0.25$. The base case holds true.

Step 2: Inductive Hypothesis

Assume that the statement holds true for $n = k$, i.e., in the interval $[k, k + 100]$

$$\Pi(k + 100) - \Pi(k) \leq 25 \quad (3)$$

This implies that the number of primes in any 100 consecutive numbers starting at k is at most 25.

Step 3: Inductive Step($n=k+1$)

We now prove that the statement holds for $n = k + 1$, i.e., in the interval $[k + 1, k + 101]$:

$$\Pi((k + 1) + 100) - \Pi(k + 1) \leq 25 \quad (4)$$

It is noted that the interval $[k + 1, k + 101]$ overlaps significantly with $[k, k + 100]$, except for the first number (k) and the last number ($k + 101$). Let us analyze this shift:

a. If k (the first number in $[k, k + 100]$) is a prime, it is no longer included in $[k + 1, k + 101]$.

b. If $k+101$ (the new number in $[k + 1, k + 101]$) is a prime, it is added to the count.

Combining Both Cases The actual change in the count of primes depends on whether k and $k + 101$ are primes. The two cases can happen simultaneously:

If k is prime but $k + 101$ is not, the count decreases by 1.

If $k + 101$ is prime but k is not, the count increases by 1.

If both are primes, the count remains the same.

If neither are primes, the count remains the same.

Thus, the change in the number of primes is at most:

$$\Pi((k + 1) + 100) - \Pi(k + 1) \leq \Pi((k + 100) - \Pi(k)) + 1 - 1 \leq 25 \quad (5)$$

This shows that the number of primes in $[k + 1, k + 101]$ cannot exceed 25, and the probability remains:

$$P = \frac{\Pi((k + 1) + 100) - \Pi(k + 1)}{100} \leq 25 \quad (6)$$

Therefore, by the principle of mathematical induction, the statement is true for all $n \geq 1$. Thus: The numbers of prime number in the set $A = \{n, n + 1, \dots, n + 99: n \in \mathbb{N}\}$ cannot be more than 25. In other words, if p denotes the probability of a prime number choosing from A than $p \leq \frac{1}{4}$.

Conclusion

By mathematical induction, we have shown that the number of primes in any interval of 100 consecutive natural numbers cannot exceed 25. Consequently, the probability of encountering a prime in such intervals is at most 0.25. This result aligns with empirical observations and theoretical expectations derived from the prime number theorem, reinforcing our understanding of prime density in finite intervals. Future work could explore similar bounds for larger or smaller intervals and examine the impact of these results on applications in cryptographic algorithms and computational number theory.

References

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